



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

p-groups with all the elements of order p in the center

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

p-groups with all the elements of order p in the center / D. BUBBOLONI; CORSI TANI G. - In: ALGEBRA COLLOQUIUM. - ISSN 1005-3867. - STAMPA. - 11(2):(2004), pp. 181-190.

Availability:

This version is available at: 2158/250003 since:

Terms of use:

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

(Article begins on next page)

p -Groups with All the Elements of Order p in the Center*

Daniela Bubboloni Gabriella Corsi Tani

DIMAD Università' di Firenze

via C. Lombroso 6/17, I-50134 Firenze, Italy

E-mail: daniela.bubboloni@dmd.unifi.it corsi@cce.unifi.it

Received 26 April 2001

Communicated by C.K. Gupta

Abstract. Resting on a suitable base of the quotients of the λ_i -series for the free groups on r generators, we get, for p odd, a class of TH- p -groups (the groups in the title) nG_r with arbitrary large derived length. We prove that every TH- p -group G with r generators and exponent p^n is a quotient of nG_r and a product of m cyclic groups, where $p^m = |\Omega_1(G)|$. At last we describe the TH- p -groups of exponent p^2 .

2000 Mathematics Subject Classification: 20D15

Keywords: p -groups, central series, free groups

1 Introduction

Let G be a p -group and p an odd prime. We denote by $\Omega_i(G)$ the subgroup of G generated by the elements of order dividing p^i , and we call G a *TH- p -group* if all its elements of order p are central, i.e., $\Omega_1(G) \leq Z(G)$. This name was introduced by the authors in [2] in acknowledgment to Thompson who first obtained some classical results for the number of generators of these groups (see [4, III, 12.2]). In [2], it was shown that several properties of the regular p -groups hold also for the class of TH- p -groups. There we characterized the TH- p -groups G with $|\Omega_1(G)| = p^2$ and exhibited some other examples of TH- p -groups obtaining only metabelian groups.

In this paper, following a suggestion of C.M. Scoppola, we construct a class of TH- p -groups nG_r (see Definition 3.1) with arbitrary large derived length (see Theorems 3.2 and 3.3).

This construction rests on the properties of the central series $\lambda_i(F_r)$ of the free group F_r on r generators and on the behaviour of a particular base

*Supported by M.U.R.S.T. and C.N.R.

of the elementary abelian quotients $\lambda_i(F_r)/\lambda_{i+1}(F_r)$ (see Theorem 2.5(c)). The explicit determination of a base for $\lambda_i(F_r)/\lambda_{i+1}(F_r)$ has interest in itself and we devote Section 2 to this, resting on methods and results in [5] and [1].

Similar goals for the central series $\kappa_n(F_r)$, the Jennings–Lazard–Zassenhaus series of G , were reached by C.M. Scoppola in [6] (compare Lemma 1.11 and Proposition 2.5 in [6] with (a) and (c) of Theorem 2.5).

In Section 4, we observe that each TH- p -group with r generators and exponent p^n is a quotient of the group nG_r (Theorem 4.1). Moreover, we obtain some new general result about the structure of TH- p -groups which turn out to be a suitable product of cyclic groups (Theorem 4.3). Finally, in Section 5, we describe the TH- p -groups of exponent p^2 .

The notation is standard. We indicate by $\gamma_i(G)$ the i -th term of the lower central series of a group G . Throughout this paper, p will be always an odd prime.

2 The λ_i -Series of the Free Groups

We recall the construction of the central series $\lambda_i(G)$ of a group G and the properties of this series which we intend to use, collecting them from [5] and [1].

Definition 2.1. For any group G , put

$$\lambda_i(G) := \gamma_1(G)^{p^{i-1}} \gamma_2(G)^{p^{i-2}} \cdots \gamma_i(G) \quad (i \geq 1).$$

Thus, $\lambda_i(G)$ is a characteristic subgroup of G and

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \cdots.$$

Theorem 2.2. [5, 1] *For any $i \in \mathbb{N}$, the following properties hold:*

- (a) $[\lambda_i(G), \lambda_j(G)] \leq \lambda_{i+j}(G)$;
- (b) $\lambda_i(G) = [\lambda_{i-1}(G), G] \lambda_{i-1}(G)^p$;
- (c) $[\lambda_i(G), G] = \gamma_2(G)^{p^{i-1}} \cdots \gamma_{i+1}(G)$;
- (d) *if $G/\gamma_j(G)$ is torsion free, then $\lambda_i(G) \cap \gamma_j(G) = \gamma_j(G)^{p^{i-j}} \cdots \gamma_i(G)$;*
- (e) $\lambda_i(G)^{p^j} \leq \lambda_{i+j}(G)$;
- (f) *the λ_i -series is central and $\lambda_i(G)/\lambda_{i+1}(G)$ is an elementary abelian p -group.*

Definition 2.3. Let F_r be the free group on r free generators x_1, x_2, \dots, x_r , and $\mathcal{A} = (a_1, a_2, \dots, a_n)$ be an ordered subset of F_r . We denote by \mathcal{A}^p the ordered subset $\mathcal{A}^p = (a_1^p, a_2^p, \dots, a_n^p)$. Moreover, if $H \trianglelefteq F_r$, we denote with $\mathcal{A} \bmod H$ the ordered subset $(a_1H, a_2H, \dots, a_nH)$ of F_r/H .

For brevity, we will often write λ_i and γ_i instead of $\lambda_i(F_r)$ and $\gamma_i(F_r)$, respectively.

Lemma 2.4. *Let $a_1, a_2, \dots, a_n \in \lambda_{i-1}$ ($i \geq 2$). Then*

$$(a_1 a_2 \cdots a_n)^p \equiv (a_1^p a_2^p \cdots a_n^p) \pmod{\lambda_{i+1}}.$$

Proof. Let $a_1, a_2 \in \lambda_{i-1}$. By the Hall–Petrescu formula, there exist elements $c_k \in \gamma_k(\lambda_{i-1})$ ($k = 2, \dots, p$) such that

$$a_1^p a_2^p = (a_1 a_2)^p c_2^{\binom{p}{2}} \cdots c_{p-1}^{\binom{p}{p-1}} c_p.$$

Since $i, k \geq 2$, we get $\lambda_{k(i-1)} \leq \lambda_i$, and by (a) and (e) of Theorem 2.2, we obtain $\gamma_k(\lambda_i) \leq \lambda_{ki}$ and $\lambda_i^p \leq \lambda_{i+1}$; so whenever $k \leq p-1$, we get

$$c_k^{\binom{p}{k}} \in \gamma_k(\lambda_{i-1})^p \leq \lambda_{k(i-1)}^p \leq \lambda_i^p \leq \lambda_{i+1}.$$

For $k = p$, since $i \geq 2$ and $p \geq 3$ imply $p(i-1) \geq i+1$, we get

$$c_p \in \gamma_p(\lambda_{i-1}) \leq \lambda_{p(i-1)} \leq \lambda_{i+1}.$$

Hence,

$$a_1^p a_2^p \equiv (a_1 a_2)^p \pmod{\lambda_{i+1}}.$$

Now by induction on n , the lemma follows at once. \square

Theorem 2.5.

- (a) λ_{i-1}/λ_i may be embedded into λ_i/λ_{i+1} . Moreover, there exists a base $\mathcal{A}_{i-1} \pmod{\lambda_i}$ of λ_{i-1}/λ_i such that $\mathcal{A}_{i-1}^p \pmod{\lambda_{i+1}}$ is independent in λ_i/λ_{i+1} .
- (b) For each i , let $\mathcal{A}_i \pmod{\gamma_{i+1}}$ be a base of γ_i/γ_{i+1} . Then

$$(\mathcal{A}_1^{p^{i-1}} \cup \mathcal{A}_2^{p^{i-2}} \cup \cdots \cup \mathcal{A}_i) \pmod{\lambda_{i+1}}$$

is a base for λ_i/λ_{i+1} .

- (c) Let \mathcal{C}_i denote the set of the basic commutators of weight i in a fixed sequence, then

$$\mathcal{B}_i := (\mathcal{C}_1^{p^{i-1}} \cup \mathcal{C}_2^{p^{i-2}} \cup \cdots \cup \mathcal{C}_i) \pmod{\lambda_{i+1}}$$

is a base of λ_i/λ_{i+1} .

- (d) The map $\varphi_i : \lambda_{i-1} \rightarrow \lambda_i/\lambda_{i+1}$ defined by $x\varphi_i = x^p \lambda_{i+1}$ is a homomorphism and $\ker \varphi_i = \lambda_i$.

Proof. (a) By a well-known result of Blackburn [5, VIII, 1.9b)], since p is odd, there is an isomorphism

$$\alpha_i : \frac{\gamma_1}{\gamma_1^p \gamma_2} \times \cdots \times \frac{\gamma_i}{\gamma_i^p \gamma_{i+1}} \longrightarrow \lambda_i/\lambda_{i+1}$$

given by $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_i)\alpha_i = a_1^{p^{i-1}} a_2^{p^{i-2}} \dots a_i \lambda_{i+1}$. If μ is the natural immersion of $\frac{\gamma_1}{\gamma_1^p \gamma_2} \times \dots \times \frac{\gamma_{i-1}}{\gamma_{i-1}^p \gamma_i}$ into $\frac{\gamma_1}{\gamma_1^p \gamma_2} \times \dots \times \frac{\gamma_i}{\gamma_i^p \gamma_{i+1}}$, then the map

$$\pi_i = \alpha_{i-1}^{-1} \mu \alpha_i : \lambda_{i-1}/\lambda_i \longrightarrow \lambda_i/\lambda_{i+1}$$

is a monomorphism.

Let $\bar{a}_j \in \frac{\gamma_j}{\gamma_j^p \gamma_{j+1}}$ and $x = (1, \dots, 1, \bar{a}_j, 1, \dots, 1)\alpha_{i-1} \in \lambda_{i-1}/\lambda_i$. Then $x = a_j^{p^{i-j-1}} \lambda_i$ and we have

$$x\pi_i = a_j^{p^{i-j}} \lambda_{i+1} \in \lambda_i/\lambda_{i+1}.$$

Now if $\mathcal{C}_j \bmod \gamma_j^p \gamma_{j+1}$ is a base of $\gamma_j/\gamma_j^p \gamma_{j+1}$, then

$$\bigcup_{j=1}^{i-1} ((\mathcal{C}_j \bmod \gamma_j^p \gamma_{j+1})\alpha_{i-1}) = \bigcup_{j=1}^{i-1} (\mathcal{C}_j^{p^{i-1-j}} \bmod \lambda_i)$$

is a base, say $\mathcal{A}_{i-1} \bmod \lambda_i$ of λ_{i-1}/λ_i . Hence, we obtain that

$$(\mathcal{A}_{i-1} \bmod \lambda_i)\pi_i = \bigcup_{j=1}^{i-1} (\mathcal{C}_j^{p^{i-j}} \bmod \lambda_{i+1}) = \mathcal{A}_{i-1}^p \bmod \lambda_{i+1}$$

is a base of $\text{Im } \pi_i$.

(b) First of all, we observe that if $\mathcal{A}_i \bmod \gamma_{i+1}$ is a base of γ_i/γ_{i+1} , since γ_i/γ_{i+1} is a torsion-free abelian group, $\mathcal{A}_i \bmod \gamma_i^p \gamma_{i+1}$ is a base of the elementary abelian group $\gamma_i/\gamma_i^p \gamma_{i+1}$.

Next the base $(\mathcal{A}_1 \bmod \gamma_1^p \gamma_2) \cup \dots \cup (\mathcal{A}_i \bmod \gamma_i^p \gamma_{i+1})$ of

$$\frac{\gamma_1}{\gamma_1^p \gamma_2} \times \dots \times \frac{\gamma_i}{\gamma_i^p \gamma_{i+1}}$$

is taken onto the base $(\mathcal{A}_1^{p^{i-1}} \cup \mathcal{A}_2^{p^{i-2}} \cup \dots \cup \mathcal{A}_i) \bmod \lambda_{i+1}$ of λ_i/λ_{i+1} by the isomorphism α_i which we mentioned in the proof of (a).

(c) By the well-known Hall's basis theorem, $\mathcal{C}_i \bmod \gamma_{i+1}$ is a base of γ_i/γ_{i+1} (see [3, Chapter 11]). Thus, (c) follows from (b).

(d) By Lemma 2.4, the map $\varphi_i : \lambda_{i-1} \rightarrow \frac{\lambda_i}{\lambda_{i+1}}$ given by $x\varphi_i = x^p \lambda_{i+1}$ is a homomorphism.

We prove $\ker \varphi_i = \lambda_i$. Clearly, $\lambda_i \leq \ker \varphi_i$, and we only need to show that for $x \in \lambda_{i-1}$, the condition $x^p \in \lambda_{i+1}$ implies $x \in \lambda_i$. By (a), we can choose a base $\mathcal{A}_{i-1} \bmod \lambda_i$ for λ_{i-1}/λ_i with $\mathcal{A}_{i-1} = (a_1, a_2, \dots, a_t)$ such that $\mathcal{A}_{i-1}^p \bmod \lambda_{i+1}$ is independent in λ_i/λ_{i+1} . Let $x \equiv a_1^{\alpha_1} a_2^{\alpha_2} \dots a_t^{\alpha_t} \bmod \lambda_i$ for suitable $0 \leq \alpha_i < p$. By Lemma 2.4, we get

$$x^p \equiv a_1^{\alpha_1 p} a_2^{\alpha_2 p} \dots a_t^{\alpha_t p} \bmod \lambda_{i+1},$$

hence, $x^p \in \lambda_{i+1}$ implies

$$a_1^{\alpha_1 p} a_2^{\alpha_2 p} \dots a_t^{\alpha_t p} \equiv 1 \bmod \lambda_{i+1}.$$

Since $\mathcal{A}_{i-1}^p \bmod \lambda_{i+1}$ is independent, we obtain $\alpha_1 = \alpha_2 = \cdots = \alpha_t = 0$. \square

Note that the monomorphism of elementary abelian groups

$$\bar{\varphi}_i : \lambda_{i-1}/\lambda_i \longrightarrow \lambda_i/\lambda_{i+1}$$

induced by φ_i coincides with the π_i introduced in the proof of (a) since they agree on a base of λ_{i-1}/λ_i .

3 The TH- p -Groups nG_r

Definition 3.1. For all positive integers n and $r \geq 2$, we set

$${}^nG_r = \frac{F_r}{\lambda_{n+1}(F_r)}.$$

Theorem 3.2. The groups nG_r have the following properties:

- (a) nG_r is a finite p -group of order $p^{|\mathcal{B}_1| + \cdots + |\mathcal{B}_n|}$.
- (b) The nilpotency class of nG_r is n .
- (c) The exponent of nG_r is p^n .
- (d) $\lambda_i({}^nG_r) = \lambda_i/\lambda_{n+1}$ for all $i \in \mathbb{N}$.
- (e) The derived length $\text{dl}({}^nG_r)$ of nG_r is $\log_2 \frac{n+1}{3} + 2$ when $\log_2 \frac{n+1}{3}$ is an integer, otherwise we have

$$\lceil \log_2 \frac{n+1}{3} \rceil + 1 \leq \text{dl}({}^nG_r) \leq \lceil \log_2 \frac{n+1}{3} \rceil + 2,$$

where $\lceil a \rceil$ denotes the upper integral part of the real number a .

Proof. (a) The order of nG_r follows immediately from Theorem 2.5(c).

(b) Since $\gamma_{n+1}({}^nG_r) = \frac{\gamma_{n+1}\lambda_{n+1}}{\lambda_{n+1}} = 1$, the nilpotency class of nG_r is at most n . On the other side, $\mathcal{C}_n \bmod \lambda_{n+1}$ generates $\gamma_n({}^nG_r) = \frac{\gamma_n\lambda_{n+1}}{\lambda_{n+1}}$, and by Theorem 2.5(c), $\mathcal{C}_n \bmod \lambda_{n+1}$ is included in a base of λ_n/λ_{n+1} , so $\gamma_n({}^nG_r)$ does not reduce to the identity.

(c) Since $F_r p^n = \lambda_1 p^n \leq \lambda_{n+1}$ by Theorem 2.2(e), we get $\exp {}^nG_r \leq p^n$. But $\mathcal{C}_1 p^{n-1} \bmod \lambda_{n+1}$ is included in a base of λ_n/λ_{n+1} , hence $\exp {}^nG_r = p^n$.

(d) We prove that $\lambda_i({}^nG_r) = \lambda_i/\lambda_{n+1}$ by induction on i . The inductive hypothesis gives

$$\lambda_{i-1}({}^nG_r) = \lambda_{i-1}/\lambda_{n+1},$$

hence by Theorem 2.2(b), we get

$$\begin{aligned} \lambda_i(F_r/\lambda_n) &= [\lambda_{i-1}(F_r/\lambda_n), F_r/\lambda_{n+1}] \lambda_{i-1}(F_r/\lambda_{n+1})^p \\ &= \left[\frac{\lambda_{i-1}}{\lambda_{n+1}}, \frac{F_r}{\lambda_{n+1}} \right] \left(\frac{\lambda_{i-1}}{\lambda_{n+1}} \right)^p = \frac{[\lambda_{i-1}, F_r]}{\lambda_{n+1}} \frac{\lambda_{i-1}^p \lambda_{n+1}}{\lambda_{n+1}} = \frac{\lambda_i}{\lambda_{n+1}}. \end{aligned}$$

This also holds for $i = 1$ by the definition of nG_r .

(e) Since ${}^nG_r^{(k)} \leq \gamma_{2^k}({}^nG_r) \leq \lambda_{2^k}({}^nG_r) = \lambda_{2^k}/\lambda_{n+1}$ by (d), we have ${}^nG_r^{(k)} = 1$ whenever $k \geq \log_2(n+1)$, hence $\text{dl}({}^nG_r) \leq \lceil \log_2(n+1) \rceil$.

Next let $x_1 < x_2 < \dots < x_r$ be the basic commutators of weight 1 in F_r . We define inductively the elements $u_i \in F_r$ by setting $u_1 = [x_2, x_1]$ and $u_{i+1} = [[u_i, x_1], u_i]$. It is easily checked by induction on i that the $u_i \in F_r^{(i)}$ are basic commutators of weight $w_i = 2^{i-1}3 - 1$. Therefore, $u_i \in \lambda_{w_i}$ and $u_i \notin \lambda_{w_i+1}$ since, by Theorem 2.5(c), the $u_i \bmod \lambda_{w_i+1}$ belong to a base of $\lambda_{w_i}/\lambda_{w_i+1}$. This means that $({}^{w_i}G_r)^{(i)} \neq 1$. Thus,

$$i < \text{dl}({}^{w_i}G_r) \leq \lceil \log_2(w_i + 1) \rceil = i + 1,$$

hence $\text{dl}({}^{w_i}G_r) = i + 1$. If $\log_2 \frac{n+1}{3} = i - 1$ is an integer, we get $n = w_i$, so $\text{dl}({}^nG_r) = \log_2 \frac{n+1}{3} + 2$.

If $\log_2 \frac{n+1}{3}$ is not an integer, there exists an integer i such that $w_i < n < w_{i+1}$. It is then clear that

$$i + 1 = \text{dl}({}^{w_i}G_r) \leq \text{dl}({}^nG_r) \leq \text{dl}({}^{w_{i+1}}G_r) = i + 2$$

and $i - 1 < \log_2 \frac{n+1}{3} < i$. The thesis follows. \square

Theorem 3.3. nG_r is a TH- p -group on r generators.

Proof. First of all, we prove that nG_r is a TH- p -group. Since the λ_i -series is central, we only need to show that $x \in F_r$ and $x^p \in \lambda_{n+1}$ imply $x \in \lambda_n$. By contradiction, assume $x \in \lambda_{i-1} - \lambda_i$ with $i \leq n$. Since $x^p \in \lambda_{n+1} \leq \lambda_{i+1}$, by Theorem 2.5(d), we get $x \in \ker \varphi_i = \lambda_i$, against the assumption.

Next we observe that, by Theorem 3.2, $\Phi({}^nG_r) = \lambda_2({}^nG_r)$ coincides with λ_2/λ_{n+1} , and therefore, by Theorem 2.5(c), $|{}^nG_r : \Phi({}^nG_r)| = |\lambda_1 : \lambda_2| = p^r$, which means $d({}^nG_r) = r$. \square

Remark 3.4. If G is a TH- p -group of exponent p^n , then $\lambda_{n+1-i}(G) \leq \Omega_i(G)$. In particular $\lambda_{n+1}(G) = 1$.

Indeed, this follows immediately from [5, VIII, 1.6] since the Ω_i -series is central with elementary abelian factors (see [2, 2.3]).

Theorem 3.5. For $1 \leq i \leq n$ and $r \geq 2$, it holds $\Omega_i({}^nG_r) = \lambda_{n+1-i}({}^nG_r) = \frac{\lambda_{n+1-i}}{\lambda_{n+1}}$. Moreover, $|\Omega_1({}^nG_r)| = p^m$, where m is the number of the basic commutators of weight at most n on r free generators.

Proof. We proceed by induction on i . It follows from the proof of Theorem 3.3 that for all $x \in F_r$, we have $x^p \in \lambda_{n+1}$ if and only if $x \in \lambda_n$. By Theorem 3.2(d), this yields $\Omega_1({}^nG_r) = \lambda_n/\lambda_{n+1} = \lambda_n({}^nG_r)$. Assume now that $\Omega_i({}^nG_r) = \lambda_{n+1-i}({}^nG_r)$. Then using [2, Lemma 2.2], Theorem 3.2(d)

and the case $i = 1$, we have

$$\begin{aligned} \frac{\Omega_{i+1}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)} &= \frac{\Omega_{i+1}({}^nG_r)}{\Omega_i({}^nG_r)} = \Omega_1\left(\frac{{}^nG_r}{\Omega_i({}^nG_r)}\right) = \Omega_1\left(\frac{\lambda_1/\lambda_{n+1}}{\lambda_{n+1-i}/\lambda_{n+1}}\right) \\ &\cong \Omega_1(\lambda_1/\lambda_{n+1-i}) = \Omega_1({}^{n-i}G_r) = \lambda_{n-i}({}^{n-i}G_r) = \lambda_{n-i}/\lambda_{n-i+1} \\ &\cong \frac{\lambda_{n-i}/\lambda_{n+1}}{\lambda_{n-i+1}/\lambda_{n+1}} = \frac{\lambda_{n-i}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)}. \end{aligned}$$

Thus,

$$\frac{\Omega_{i+1}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)} \cong \frac{\lambda_{n-i}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)},$$

and since $\lambda_{n-i}({}^nG_r) \leq \Omega_{i+1}({}^nG_r)$ by Remark 3.4, we get that $\Omega_{i+1}({}^nG_r) = \lambda_{n-i}({}^nG_r)$.

Finally, by Theorem 3.2(d), we have $|\Omega_1({}^nG_r)| = |\lambda_n({}^nG_r)| = |\lambda_n/\lambda_{n+1}|$, and Theorem 2.5(c) says that there is a base for the elementary abelian group λ_n/λ_{n+1} which consists of as many elements as the number of the basic commutators of weight at most n . \square

4 TH- p -Groups

In this section, we want to describe deeply the structure of the TH- p -groups.

Theorem 4.1. *Let G be a TH- p -group with r generators and $\exp G = p^n$. Then G is a quotient ${}^nG_r/N$, where ${}^nG_r p^{n-1} \not\leq N \leq \Phi({}^nG_r)$.*

Proof. Let F_r be the free group on r free generators and $\psi : F_r \rightarrow G$ an epimorphism. We only need to show $\ker \psi \geq \lambda_{n+1}(F_r)$. Since ψ is a homomorphism, we clearly have $\psi(\lambda_i(F_r)) \leq \lambda_i(G)$ for all i , and in particular, by Remark 3.4, we get $\psi(\lambda_{n+1}(F_r)) \leq \lambda_{n+1}(G) = 1$. \square

Remark 4.2. Let ${}^nG_r/N$ be a TH- p -group. Then ${}^nG_r/(N \cap \Omega_i({}^nG_r))$ is also a TH- p -group.

In fact, let $\bar{x} = x\lambda_{n+1} \in {}^nG_r$ for some $x \in F_r$, and let $\bar{x}(N \cap \Omega_i({}^nG_r))$ be an element of order p of ${}^nG_r/(N \cap \Omega_i({}^nG_r))$. Then $\bar{x}^p \in N \cap \Omega_i({}^nG_r) = N \cap \lambda_{n+1-i}({}^nG_r)$. In particular, \bar{x}^p is in N , and since ${}^nG_r/N$ is a TH- p -group, this implies $[\bar{x}, {}^nG_r] \leq N$. On the other side, from $\bar{x}^p \in \lambda_{n+1-i}({}^nG_r) = \lambda_{n+1-i}/\lambda_{n+1}$, it follows that $x^p \in \lambda_{n+1-i}$. By the argument used in the proof of Theorem 3.3, we get $x \in \lambda_{n-i}$. Then $[x, F_r] \leq \lambda_{n+1-i}$ and this implies $[\bar{x}, {}^nG_r] \in \lambda_{n+1-i}({}^nG_r) = \Omega_i({}^nG_r)$. Thus, $[\bar{x}, {}^nG_r] \leq N \cap \Omega_i({}^nG_r)$, i.e., $\bar{x}(N \cap \Omega_i({}^nG_r))$ is central in ${}^nG_r/(N \cap \Omega_i({}^nG_r))$.

Theorem 4.3. *Let G be a TH- p -group with $|\Omega_1(G)| = p^m$. Then G is a product of (not in general pairwise permutable) cyclic groups $\langle a_1 \rangle, \dots, \langle a_m \rangle$, where $|a_i| \geq |a_{i+1}|$ and $\langle a_i \rangle \cap [\langle a_1 \rangle \cdots \langle a_{i-1} \rangle \langle a_{i+1} \rangle \cdots \langle a_m \rangle] = 1$ for $1 \leq i \leq m$.*

Proof. We use induction on the exponent p^n of G . When $n = 1$, G is an elementary abelian p -group and the theorem holds. Next we observe that $G/\Omega_1(G)$ is a TH- p -group of exponent p^{n-1} . Setting

$$p^{m'} = \left| \Omega_1 \left(\frac{G}{\Omega_1(G)} \right) \right| = \left| \frac{\Omega_2(G)}{\Omega_1(G)} \right|,$$

by Theorem 2.3 in [2], we have $m' \leq m$. By the inductive hypothesis, we have $\frac{G}{\Omega_1(G)} = \langle \bar{a}_1 \rangle \cdots \langle \bar{a}_{m'} \rangle$, where for $1 \leq i \leq m'$, $|\bar{a}_i| \geq |\bar{a}_{i+1}|$ and $\langle \bar{a}_i \rangle \cap [\langle \bar{a}_1 \rangle \cdots \langle \bar{a}_{i-1} \rangle \langle \bar{a}_{i+1} \rangle \cdots \langle \bar{a}_{m'} \rangle] = 1$.

Now we set $|\bar{a}_i| = |\Omega_1(G)| = p^{h_i}$ and $H = \langle a_1^{p^{h_1}}, \dots, a_{m'}^{p^{h_{m'}}} \rangle \leq \Omega_1(G)$.

We claim that $H = \langle a_1^{p^{h_1}} \rangle \times \cdots \times \langle a_{m'}^{p^{h_{m'}}} \rangle$. Assume the contrary, then for some i and integers $0 \leq x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m'} < p$, we have

$$a_i^{p^{h_i}} = a_1^{x_1 p^{h_1}} \cdots a_{i-1}^{x_{i-1} p^{h_{i-1}}} a_{i+1}^{x_{i+1} p^{h_{i+1}}} \cdots a_{m'}^{x_{m'} p^{h_{m'}}}.$$

Let

$$b = a_i^{-p^{h_i-1}} a_1^{x_1 p^{h_1-1}} \cdots a_{i-1}^{x_{i-1} p^{h_{i-1}-1}} a_{i+1}^{x_{i+1} p^{h_{i+1}-1}} \cdots a_{m'}^{x_{m'} p^{h_{m'}-1}},$$

and observe that for all j , $a_j^{-p^{h_j-1}} \in \Omega_2(G)$, which is nilpotent of class 2. Then the Hall–Petrescu formula implies $b^p = 1$, so $b \in \Omega_1(G)$, a contradiction. Hence, the thesis follows immediately if $m' = m$. If $m' < m$, since $\Omega_1(G) \leq Z(G)$ is elementary abelian, there exists a subgroup $K = \langle a_{m'+1} \rangle \times \cdots \times \langle a_m \rangle$ of $\Omega_1(G)$ with $\Omega_1(G) = H \times K$. Then $G = \langle a_1 \rangle \cdots \langle a_m \rangle$, and the conditions in the theorem are clearly satisfied. \square

With the same argument, we can prove the following:

Theorem 4.4. *Let G be a TH- p -group with exponent p^n . If $\mathcal{A} \bmod \Omega_{n-i}(G)$ is a base of $\Omega_{n-i+1}(G)/\Omega_{n-i}(G)$, then $\mathcal{A}^p \bmod \Omega_{n-i-1}(G)$ is independent in $\Omega_{n-i}(G)/\Omega_{n-i-1}(G)$.*

Remark 4.5. Let G be a TH- p -group and $N \triangleleft G$ with $N \cap G^p = 1$. Then G/N is a TH- p -group. Actually, in this case, all the elements of order p in G/N are images of elements of order p in G .

Remark 4.6. Let G/N be a TH- p -group. If there exists an element $x \in G - \Phi(G)$ with $x^p \in N$, then G/N has a cyclic direct factor.

5 TH- p -Groups of Exponent p^2

Disregarding abelian TH- p -groups and TH- p -groups which differ from one another in a cyclic direct factor, we have

Lemma 5.1. *The TH- p -groups of exponent p^2 with r generators are precisely 2G_r and the quotients $G = {}^2G_r/N$, where $N \not\leq {}^2G'_r$, $N \leq \Phi({}^2G_r) = \Omega_1({}^2G_r)$, and $N \cap {}^2G_r^p = 1$.*

Proof. The third condition guarantees that G is a TH- p -group (see Remark 4.5). Conversely, if G is a non-abelian TH- p -group, then the first condition is fulfilled, the second one follows from Theorem 4.1, and the third one is a consequence of Remark 4.6. \square

Now we study in more details the case $r = 3$. We get

$${}^2G_3 = \langle x, y, z \mid x^{p^2} = y^{p^2} = z^{p^2} = [y, x]^p = [z, x]^p = [z, y]^p = 1 \rangle.$$

Let $H = \langle x^p, y^p, z^p \rangle = {}^2G_3^p$ and $K = \langle [y, x], [z, x], [z, y] \rangle = {}^2G_3'$. By Lemma 5.1, the relevant TH- p -group quotients are obtained by $N \leq \Omega_1({}^2G_3) = H \times K$ with $N \cap H = 1$ and $N \cap K \neq K$. From the condition $N \cap H = 1$, it follows that the allowed orders for N are p, p^2, p^3 . The possible orders for the commutator subgroup in the correspondent TH- p -groups are p^2 or p^3 in the first case; p, p^2 or p^3 in the others two cases. Thus, we get at least eight non-isomorphic TH- p -groups. But we may observe that there are non-isomorphic TH- p -groups with the same order and the same commutator subgroup order, as we see analyzing the case $|N| = p$.

Remark 5.2. (a) If $N < K$, then $G = {}^2G_3/N$ is isomorphic to the group $G_1 = \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = [w, v] = 1 \rangle$.

(b) If $N \cap K = 1$, then $G = {}^2G_3/N$ is isomorphic to one of the following non-isomorphic groups:

$$\begin{aligned} G_2 &= \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = 1, [w, v] = u^{-p} \rangle, \\ G_3 &= \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = 1, [w, v] = w^{-p} \rangle. \end{aligned}$$

For (a), first of all, we observe that every $k = [y, x]^a [z, x]^b [z, y]^c \in K$ is a commutator. Namely, as 2G_3 is nilpotent of class 2, if $c \not\equiv 0 \pmod{p}$, we have $k = [z^c x^{-a}, y x^{bc-1}]$; while if $c \equiv 0 \pmod{p}$, we have $k = [y^a z^b, x]$. Let $N = \langle k \rangle$, where $k = [g_2, g_1]$. Since $\Phi({}^2G_3) = Z({}^2G_3)$, we see that $g_1 \Phi({}^2G_3)$ and $g_2 \Phi({}^2G_3)$ are independent, and one among the elements x, y, z , say x , does not belong to the subgroup $\langle g_1, g_2, \Phi({}^2G_3) \rangle$. Thus, g_1, g_2, x constitute a base of 2G_3 , and the map $x \mapsto x, y \mapsto g_1, z \mapsto g_2$ extends to an automorphism of 2G_3 which maps $N_0 = \langle [z, y] \rangle$ onto the subgroup $N = \langle k \rangle$.

In the case (b), the group N is generated by an element of the form $[y, x]^a [z, x]^b [z, y]^c x^{dp} y^{ep} z^{fp}$, where not all of a, b, c and not all of d, e, f are zero. As we showed in (a), the element $[y, x]^a [z, x]^b [z, y]^c$ is a commutator $[g_2, g_1]$, and it is convenient to separate two cases according to $g_3 = x^d y^e z^f \notin \langle g_1, g_2 \rangle$ or $g_3 = x^d y^e z^f \in \langle g_1, g_2 \rangle$.

Assume $g_3 = x^d y^e z^f \notin \langle g_1, g_2 \rangle$. The elements g_1, g_2, g_3 constitute a base for 2G_3 , and the automorphism determined by $x \mapsto g_3, y \mapsto g_2, z \mapsto g_1$ takes $N_0 = \langle [z, y] x^p \rangle$ onto N . It follows that ${}^2G_3/N \cong G_2$.

Assume $g_3 = x^d y^e z^f \in \langle g_1, g_2 \rangle$. Then by the explicit form of g_1, g_2 given in (a), we get either $g_3 \in \langle z^{cp} x^{-ap}, y^p x^{bc-1p} \rangle$ or $g_3 \in \langle y^{ap} z^{bp}, x^p \rangle$. Anyway,

this implies

$$be = cd + af. \quad (*)$$

We show that $(*)$ allows to define an automorphism of 2G_3 which takes the element $[z, y]z^p$ into the element $[y, x]^a[z, x]^b[z, y]^c x^{dp} y^{ep} z^{fp}$. Namely, if $e, f \not\equiv 0 \pmod{p}$, this automorphism is determined by $x \mapsto x$, $y \mapsto x^{e^{-1}(dcf^{-1}+a)} y^{cf^{-1}}$, $z \mapsto x^d y^e z^f$; while if $f \equiv 0$ and $e \not\equiv 0$, it is determined by $x \mapsto x$, $y \mapsto x^{ae^{-1}} z^{-ce^{-1}}$, $z \mapsto x^a y^e z^f$; and finally, if $f \not\equiv 0$ and $e \equiv 0$, it is determined by $x \mapsto x$, $y \mapsto x^{bf^{-1}} y^{cf^{-1}}$, $z \mapsto x^d y^e z^f$. Observe that, by $(*)$, the only remaining possibility is $c, e, f \equiv 0$, and in this case, the automorphism is given by $x \mapsto x$, $y \mapsto y^{-ad^{-1}} z^{-bd^{-1}}$, $z \mapsto x^d y^e z^f$. It follows that ${}^2G_3/N \cong G_3$.

Finally, it is easily checked directly that there is no isomorphism between G_2 and G_3 .

Acknowledgement. We wish to thank Michael Vaughan Lee for many enlightening remarks.

References

- [1] N. Blackburn, L. Evens, Schur multipliers of p -groups, *J. Reine Angew. Math.* **309** (1979) 100–113.
- [2] D. Bubboloni, G. Corsi Tani, p -Groups with some regularity properties, *Ric. di Mat.* **49** (2) (2000) 327–339.
- [3] M. Hall, *The Theory of Groups*, Chelsea, New York, 1976.
- [4] B. Huppert, *Endliche Gruppen*, Vol. 1, Springer-Verlag, New York-Berlin-Heidelberg, 1967.
- [5] B. Huppert, N. Blackburn, *Finite Groups*, Vol. II, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
- [6] C.M. Scoppola, Groups of prime power order as Frobenius–Wielandt complements, *Trans. Amer. Math. Soc.* **325** (1991) 855–874.